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The quantum fluctuations of (1+1)-dimensional real scalar fields around solitons at the finite temperatures

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Abstract. We investigate the quantum fluctuations of real scalar fields in a (1+1)-dimensional space around the solitons at the finite temperatures, using a method based on the real-time Green function approach. We calculate the temperature dependence of the soliton masses in a sine-Gordon system and a ϕ^4 system respectively. We have taken into account the corrections due to the existence of bound states and the phase shifts for continuum states in soliton potentials.

1. Introduction

In a previous paper, Su *et al* (1983, hereafter referred to as I), presented a method based on the concept of the coherent state and the approach of the real-time Green function to investigate the spontaneous breaking of symmetry in the (1+1)-dimensional ϕ^4 field, as well as its restoration at finite temperatures. It was found that the soliton solution, being a coherent state, obeys a 'classical' field equation with a temperature dependent parameter M^2 instead of a fixed one, (see I (5.7)). Further quantum fluctuations are excited on this background and handled by the Green function method. Therefore, in this paper, we shall take a more straightforward approach by assuming that the (1+1)-dimensional quantum field ϕ is composed of two parts.

$$\hat{\phi} = \phi_s + \hat{\psi} \tag{1.1}$$

(Dashen *et al* 1974, Maki and Takayama 1979a, b) where ϕ_s is the 'classical' soliton solution while $\hat{\psi}$ is the fluctuation around it. The latter is quantised as follows:

$$\hat{\psi}(x,t) = \sum_k \frac{1}{(2L\omega_k)^{1/2}} (c_k(t) + c^+_{-k}(t)) \exp(ikx) \tag{1.2}$$

where

$$\omega_k = (\mu^2 + k^2)^{1/2} \tag{1.3}$$

with μ remains arbitrary and will be used as a tool in calculation as discussed in I. Furthermore, aiming at carefully studying the quantum fluctuations around the soliton, we will introduce the momentum non-conserved Green function and try to make the theory more complete.

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The organisation of this paper is as follows. Firstly in § 2 we study the sine-Gordon system and carry out a μ normal-ordering renormalisation scheme. Then in § 3 we reduce the Hamiltonian in such a way that we are able to present a refined version of the real-time Green function approach in § 4. Accordingly, in § 5 we discuss the problem about the existence of a critical temperature T_c in the SG system. In § 6 we re-examine the ϕ^4 system. Section 7 is the summary and discussion.

2. μ normal-ordering and renormalisation

We begin with the Lagrangian density of sine–Gordon system

$$\mathscr{L} = \tfrac{1}{2}(\partial\phi/\partial t)^2 - \tfrac{1}{2}(\partial\phi/\partial x)^2 + (m_b^2/g^2)\cos g\phi - m_b^2/g^2 \tag{2.1}$$

and try to make a transition from classical theory to quantum theory by quantising the ϕ field in the Heisenberg picture:

$$\phi(x,t) = \sum_k (2L\omega_k)^{-1/2}(a_k(t)\exp(\mathrm{i}kx) + a_k^{\dagger}(t)\exp(-\mathrm{i}kx)) \tag{2.2a}$$

$$\pi(x,t) = \partial\phi/\partial t = \sum_k \mathrm{i}\,(\omega_k^{1/2}/2L)(a_k^{\dagger}(t)\exp(-\mathrm{i}kx) - a_k(t)\exp(\mathrm{i}kx)) \tag{2.2b}$$

with

$$[a_k(t), a_{k'}^{\dagger}(t)] = \delta_{kk'} \tag{2.3}$$

and

$$\omega_k = (\mu^2 + k^2)^{1/2} \tag{2.4}$$

where the mass μ remains arbitrary at the present stage. In order to eliminate the ambiguities stemming from the ultraviolet divergences in quantum field calculations, a normal ordering procedure is necessary. All the operators $a^{\dagger}$ will be rearranged on the left while all the a's will be on the right, for example,

$$N_\mu(a_k a_k^{\dagger}) \equiv :a_k a_k^{\dagger}: \equiv a_k^{\dagger} a_k. \tag{2.5}$$

By means of the Baker–Campbell–Hausdorff theorem

$$\mathrm{e}^{A+B} = \mathrm{e}^A \mathrm{e}^B \exp(-1/2[A,B]) \tag{2.6}$$

where $[A, B]$ commutes with A and B respectively, we have

$$\exp(\mathrm{i}g\phi) = (\mu^2/4\Lambda^2)^{g^2/8\pi} N_\mu[\exp(\mathrm{i}g\phi)] \tag{2.7}$$

with Λ being the cut-off momentum. Therefore, we get from (2.1) the Hamiltonian density

$$\mathscr{H} = N_\mu[(\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\partial\phi/\partial x)^2 - (m^2/g^2)\cos g\phi + D_0] \tag{2.8}$$

where

$$m^2 = m_b^2(\mu^2/4\Lambda^2)^{g^2/8\pi} = m_b^2\exp[(g^2/4\pi)\ln(\mu/2\Lambda)] \tag{2.9}$$

$$D_0 = m_b^2/g^2 + (8\pi)^{-1}\int \mathrm{d}k\,(2\omega_k - \mu^2/\omega_k). \tag{2.10}$$

The parameters m^2 and g^2 are all finite numbers now. Notice, however, that m^2 is μ dependent. The fact that the normal-ordering prescription with respect to a mass

μ is equivalent to a renormalisation in parameters was proved by Coleman (1975) and Chang (1976). We will choose μ as the mass of the elementary excitation (phonon) in our system.

3. The reduced Hamiltonian of the sine–Gordon system

Starting from the renormalisation Hamiltonian density (2.8), we substitute (1.1) into it and quantise the fluctuation $\hat{\psi}$ as (1.2). Having the experience of ϕ_s as a time-independent classical field which can commute freely with the operators c. The mass parameter μ in (1.3) is also the same as in (2.4). Since

$$N_\mu \cos g\phi = N_\mu \cos g(\phi_s+\hat{\psi}) = N_\mu(\cos g\phi_s \cos g\hat{\psi} - \sin g\phi_s \sin g\hat{\psi}) \tag{3.1}$$

by expanding $\cos g\hat{\psi}$ and $\sin g\hat{\psi}$ in terms of the operators c the Hamiltonian will be

$$H = H_s + H_1 + H_1' + H_2 + H_2' + H_3 + H_4 + \dots \tag{3.2}$$

where

$$H_s = \int [\tfrac{1}{2}(\partial\phi_s/\partial x)^2 - (m^2/g^2)\cos g\phi_s + D_0]\,dx \tag{3.3}$$

$$H_1 = \frac{m^2}{g}\sum_k \frac{1}{(2L\omega_k)^{1/2}} \int \sin g\phi_s\, e^{ikx}\, dx\,(c_k + c^{+}_{-k}) \tag{3.4}$$

$$H_1' = \int (\partial\phi_s/\partial x)(\partial\hat{\psi}/\partial x)\, dx \tag{3.5}$$

$$H_2' = \sum_k \left[\frac{1}{2}\left(\frac{k^2}{\omega_k} + \omega_k\right) c^{+}_k c_k + \frac{1}{4}\left(\frac{k^2}{\omega_k} - \omega_k\right)(c^{+}_k c^{+}_{-k} + c_k c_{-k})\right] \tag{3.6}$$

$$H_2 = \frac{m^2}{4L}\sum_{k_1,k_2} (\omega_{k_1}\omega_{k_2})^{-1/2} \int \cos g\phi_s \exp[i(k_1+k_2)x]\colon (c_{k_1}+c^{+}_{-k_1})(c_{k_2}+c^{+}_{-k_2})\colon \dots \tag{3.7}$$

Evidently, there are n operator products in H_n and n goes to infinity. Let us now make a pairing approximation as in I, for example,

$$c^{+}_{-k_1}c_{k_2}c_{k_3} \to \langle c^{+}_{-k_1}c_{k_2}\rangle\delta_{k_1-k_2}c_{k_3} + \langle c^{+}_{-k_1}c_{k_3}\rangle\delta_{k_1-k_3}c_{k_2} + \langle c_{k_2}c_{k_3}\rangle\delta_{k_2-k_3}c^{+}_{-k_1} \tag{3.8}$$

with $\langle c^{+}_k c_k\rangle$ and $\langle c^{+}_k c^{+}_{-k}\rangle$ $(=\langle c_{-k}c_k\rangle)$ being the ensemble average value of relevant operators. Following this pattern, we are able to reduce the whole Hamiltonian to

$$\tilde{H} = H_s + \tilde{H}_1 + \tilde{H}_2 + H_2' \tag{3.9}$$

where

$$\tilde{H}_1 = \sum_k \frac{1}{(2L\omega)^{1/2}} \int dx\, e^{ikx}\left[\frac{M^2}{g}\sin g\phi_s - \frac{d^2\phi_s}{dx^2}\right](c_k + c^{+}_{-k}) \tag{3.10}$$

$$\tilde{H}_2 = \frac{M^2}{4L}\sum_{k_1k_2}\frac{1}{(\omega_{k_1}\omega_{k_2})^{1/2}}\int \cos g\phi_s \exp[i(k_1+k_2)x]\,dx\,(c_{k_1}c_{k_2} + c^{+}_{-k_1}c^{+}_{-k_2} + 2c^{+}_{-k_1}c_{k_2}) \tag{3.11}$$

with

$$M^2 = m^2/(1+g^2\nu) \tag{3.12}$$

and

$$\nu = \sum_p \frac{1}{2L\omega_p} \langle c^+_p c_p + c^+_{-p} c^+_p \rangle \tag{3.13}$$

being temperature dependent.

Now let us evaluate the expectation value of the Hamiltonian:

$$U \equiv \langle \tilde{H} \rangle \tag{3.14}$$

and make a further pairing approximation in $\langle \tilde{H}_2 \rangle$. Then use of the variation principle

$$\delta U / \delta \phi_s(x) = 0 \tag{3.15}$$

will lead to

$$\mathrm{d}^2 \phi_s / \mathrm{d}x^2 - (M^2/g) \sin g\phi_s = 0 \tag{3.16}$$

which can also be obtained by the condition

$$\tilde{H}_1 = 0. \tag{3.17}$$

The solution of equation (3.16) is known as a kink-like soliton:

$$\phi_s(x) = (4/g) \tan^{-1} \mathrm{e}^{Mx} \tag{3.18}$$

4. A refined formalism for the real-time Green function approach

As we wish to study the quantum fluctuations carefully in the presence of a soliton state, we introduce the momentum non-conserved Green functions as follows:

$$\begin{aligned} G_1(p,q) &\equiv \langle\langle c_p | c^+_q \rangle\rangle = -\mathrm{i}\langle T c_p(t) c^+_q(t') \rangle \\ G_2(p,q) &\equiv \langle\langle c^+_{-p} | c^+_q \rangle\rangle = -\mathrm{i}\langle T c^+_{-p}(t) c^+_q(t') \rangle. \end{aligned} \tag{4.1}$$

The equation of motion for G_σ ($\sigma = 1, 2$) are

$$\begin{aligned} \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} G_1 &= \delta(t-t')\delta_{pq} + \langle\langle [c_p(t), \tilde{H}(t)|] c^+_q(t') \rangle\rangle \\ \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} G_2 &= \langle\langle [c^+_{-p}(t), \tilde{H}(t)] | c^+_q(t') \rangle\rangle. \end{aligned} \tag{4.2}$$

Making a Fourier transformation

$$\tilde{G}_\sigma(p,q) \equiv \tilde{G}_\sigma(p,q,E) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, G_\sigma(p,q) \exp[\mathrm{i}E(t-t')] \tag{4.3}$$

and using (3.9), we are able to recast equation (4.2) into

$$\begin{aligned} E\tilde{G}_1(p,q) &= (1/2\pi)\delta_{pq} + (p^2/2\omega_p + \omega_p/2)\tilde{G}_1(p,q) + (p^2/2\omega_p - \omega_p/2)\tilde{G}_2(p,q) \\ &\quad + \frac{M^2}{2L\sqrt{\omega_p}} \sum_k \frac{1}{\sqrt{\omega_k}} \int \mathrm{d}x \exp[(\mathrm{i}(k-p)x] \cos g\phi_s(\tilde{G}_1(k,q) + \tilde{G}_2(k,q)) \\ E\tilde{G}_2(p,q) &= -(p^2/2\omega_p + \omega_p/2)\tilde{G}_2(p,q) - (p^2/2\omega_p - \omega_p/2)\tilde{G}_1(p,q) \\ &\quad - \frac{M^2}{2L\sqrt{\omega_p}} \sum_k \frac{1}{\sqrt{\omega_k}} \int \mathrm{d}x \exp[\mathrm{i}(k-p)x] \cos g\phi_s(\tilde{G}_1(k,q) + \tilde{G}_2(k,q)). \end{aligned} \tag{4.4}$$

Noting that

$$\cos g\phi_s = 1 - 2\,\mathrm{sech}^2 Mx \tag{4.5}$$

and denoting

$$\tilde{A}(p,q) \equiv \tilde{G}_1(p,q) + \tilde{G}_2(p,q) \qquad \tilde{B}(p,q) \equiv \tilde{G}_1(p,q) - \tilde{G}_2(p,q) \tag{4.6}$$

we derive from (4.4)

$$\tilde{B}(p,q) = (1/\omega_p)[E\tilde{A}(p,q) - (1/2\pi)\delta_{pq}] \tag{4.7}$$

$$(E^2 - p^2 - M^2)\tilde{A}(p,q) + \frac{\sqrt{\omega_p}}{L}\sum_k \frac{1}{\sqrt{\omega_k}} \frac{2\pi(k-p)}{\sinh[\pi(k-p)/2M]}\tilde{A}(k,q) = \frac{\delta_{pq}}{2\pi}(E+\omega_p). \tag{4.8}$$

Denoting further

$$\tilde{F}(p,q) = L\tilde{A}(p,q)/\sqrt{\omega_p} \tag{4.9}$$

we get from (4.8)

$$(E^2 - P^2 - M^2)\tilde{F}(p,q) + \int \mathrm{d}k \frac{(p-k)}{\sinh[\pi(p-k)/2M]}\tilde{F}(k,q) = \frac{1}{\sqrt{\omega_p}}(E+\omega_p)\delta(p-q). \tag{4.10}$$

After performing a new Fourier transformation

$$\tilde{F}(p,q) = \frac{1}{2\pi}\int F(x)\exp(-\mathrm{i}px)\,\mathrm{d}x \tag{4.11}$$

we then recast the integral equation (4.10) into a non-homogeneous differential equation

$$\mathrm{d}^2F(x)/\mathrm{d}x^2 + (2M^2\,\mathrm{sech}^2 Mx - M^2 + E^2)F(x) = (\omega_q)^{1/2}(E+\omega_q)\exp(\mathrm{i}qx) \equiv S(x). \tag{4.12}$$

According to the theory of linear differential equations, the solution of (4.12) can be expressed in the following form

$$F(x) = \int K(x,x')S(x')\,\mathrm{d}x'. \tag{4.13}$$

The (mathematical) Green function $K(x,x')$ satisfies

$$(\mathscr{E} - \hat{H}(x))K(x,x') = \delta(x-x') \tag{4.14}$$

with

$$\hat{H} = -\mathrm{d}^2/\mathrm{d}x^2 - 2M^2\,\mathrm{sech}^2 Mx \tag{4.15}$$

$$\mathscr{E} \equiv E^2 - M^2 \tag{4.16}$$

and can be expanded in terms of the complete set of eigenfunctions $u_n(x)$ as follows:

$$K(x,x') = \sum_n \frac{u_n(x)u_n^*(x')}{\mathscr{E} - \mathscr{E}_n} \tag{4.17}$$

where

$$(\mathscr{E}_n - \hat{H}(x))u_n(x) = 0 \tag{4.18}$$

and

$$\int u_n^*(x)u_m(x)\,\mathrm{d}x=\delta_{mn}. \tag{4.19}$$

Substituting (4.17) into (4.13), (4.11) through (4.9) and (4.7), we finally obtain

$$\tilde{A}(p,q)=\frac{2\pi}{L}\sum_n\left(\frac{\omega_p}{\omega_q}\right)^{1/2}\frac{E+\omega_q}{E^2-E_n^2}\tilde{u}_n(p)\tilde{u}_n^*(q) \tag{4.20}$$

$$\tilde{B}(p,q)=\frac{2\pi}{L}\sum_n\frac{1}{(\omega_p\omega_q)^{1/2}}\frac{E\omega_q+E_n^2}{E^2-E_n^2}\tilde{u}_n(p)\tilde{u}_n^*(q) \tag{4.21}$$

where

$$\tilde{u}_n(p)=\frac{1}{2\pi}\int u_n(x)\exp(-\mathrm{i}px)\,\mathrm{d}x. \tag{4.22}$$

Therefore, the correlation function which we are seeking can be expressed as

$$\begin{aligned}\langle c_q^\dagger c_p+c_q^\dagger c_{-p}^\dagger\rangle &=\mathrm{i}\int\frac{\tilde{A}(p,q,E+\mathrm{i}\varepsilon)-\tilde{A}(p,q,E-\mathrm{i}\varepsilon)}{\exp(\beta E)-1}\,\mathrm{d}E\\ &=\frac{(2\pi)^2}{L}\left(\frac{\omega_p}{\omega_q}\right)^{1/2}\sum_{E_n>0}\left(\frac{\frac{1}{2}(1+\omega_q/E_n)}{\exp(\beta E_n)-1}+\frac{\frac{1}{2}(1-\omega_q/E_n)}{\exp(-\beta E_n)-1}\right)\tilde{u}_n(p)\tilde{u}_n^*(q)\end{aligned} \tag{4.23}$$

$$\begin{aligned}\langle c_q^\dagger c_p-c_q^\dagger c_{-p}^\dagger\rangle &=\mathrm{i}\int\frac{\tilde{B}(p,q,E+\mathrm{i}\varepsilon)-\tilde{B}(p,q,E-\mathrm{i}\varepsilon)}{\exp(\beta E)-1}\,\mathrm{d}E\\ &=\frac{(2\pi)^2}{L}\left(\frac{\omega_q}{\omega_p}\right)^{1/2}\sum_{E_n>0}\left[\frac{\frac{1}{2}(1+E_n/\omega_q)}{\exp(\beta E_n)-1}+\frac{\frac{1}{2}(1-E_n/\omega_q)}{\exp(-\beta E_n)-1}\right]\tilde{u}_n(p)\tilde{u}_n^*(q)\end{aligned} \tag{4.24}$$

5. Does the critical temperature of the sine–Gordon system exist?

Let us write equation (4.18) for the sine–Gordon system in the explicit form:

$$(\mathscr{E}_n+\mathrm{d}^2/\mathrm{d}x^2+2M^2\operatorname{sech}^2 Mx)u_n(x)=0. \tag{5.1}$$

Equation (5.1) can be viewed as a stationary Schrödinger equation of 'particle' with 'mass' $\frac{1}{2}$ moving in a 'potential well' $-2\mathrm{M}^2\operatorname{sech}^2 Mx$. The solutions of equation (5.1) had been discussed by Morse and Feshbach (1953). There is only one bound state with

$$\mathscr{E}_0=-M^2 \tag{5.2}$$

which corresponds to

$$E_0^2=\mathscr{E}_0+M^2=0 \tag{5.3}$$

and the eigenfunction

$$u_0(x)=(M/2)^{1/2}\operatorname{sech} Mx \tag{5.4}$$

is just the well known zero mode of the soliton. It is intimately related to the wavefunction of the soliton in configuration space through

$$u_0(x) \sim \mathrm{d}\phi_s(x)/\mathrm{d}x \tag{5.5}$$

as is easily seen by (3.18) and (5.4). Usually, the relation (5.5) is regarded as the starting point in discussing the zero mode; one also finds this mode $u_0(x)$ by solving the eigenequation

$$\omega_n^2 u_n(x) = -\partial^2/\partial x^2 u_n(x) + V''(\phi_s)u_n(x) \tag{5.6}$$

with zero eigenvalue of energy (Dashen *et al* 1974, Jackiw 1977, Gervais 1976, Rajaraman 1975, Coleman 1977, Maki and Takayama 1979). It is interesting here that equation (5.6) just coincides with (5.1) which emerges from the Green function approach with the mass parameter m in potential $V(\phi_s)$ replaced by M. Since the zero mode represents the translational degree of freedom of the soliton and has no dynamical effect, we will discard it in the latter calculation.

There are many unbound solutions of equation (5.1) with continuum spectrum describing the phonon states (see I)

$$\mathscr{E}_k^2 = k^2, \qquad \text{i.e.,} \qquad E_k^2 = M^2 + k^2, \tag{5.7}$$

where k is a continuous momentum variable. Each of these solutions describes a wave coming from the left and suffereing a phase shift η_k but without any reflection.

$$\eta_k = 2\tan^{-1} M/k. \tag{5.8}$$

The wavefunction can be expressed exactly as

$$u_k(x) = N_k \exp(\mathrm{i}kx)(k + \mathrm{i}M \tanh Mx) \tag{5.9}$$

(Naki and Takayama 1979, Rubinstein 1970). The normalisation condition (4.19) leads to

$$N_k^2 = [(k^2+M^2)L - 2M]^{-1}. \tag{5.10}$$

Substituting the Fourier transform of $u_k(x)$

$$\tilde{u}_k(p) = \frac{1}{2\pi}\int u_k(x)\exp(-\mathrm{i}px)\,\mathrm{d}x = \frac{N_k}{2}\left(2k\delta(k-p) - \frac{1}{\sinh[(k-p)\pi/2M]}\right) \tag{5.11}$$

into (4.20), we should pay attention to the boundary condition

$$k_n L + \eta_{k_n} = 2\pi n \tag{5.12}$$

and replace the summation by integration as follows

$$\sum_n \to \frac{1}{2\pi}\int \mathrm{d}k\left(L + \frac{\mathrm{d}\eta_k}{\mathrm{d}k}\right) = \frac{1}{2\pi}\int \mathrm{d}k\left(\frac{L(M^2+k^2)-2M}{(M^2+k^2)}\right). \tag{5.13}$$

Substituting (5.11) into (4.23) and putting $q = p$, we have

$$\begin{aligned}\langle c_p^+ c_p + c_{-p}^+ c_p^+\rangle &= \frac{\pi}{2L}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{k^2+M^2}\left(\frac{\frac{1}{2}(1+\omega_p/E_k)}{\exp(\beta E_k)-1} + \frac{\frac{1}{2}(1-\omega_p/E_k)}{\exp(-\beta E_k)-1}\right)\\ &\quad\times\left(\frac{L}{2\pi}4k^2\delta(k-p) + \frac{1}{\sinh^2[(k-p)\pi/2M]}\right)\end{aligned} \tag{5.14}$$

where the cross terms of $\tilde{u}_k(p)\tilde{u}_k(p)$ vanish because

$$\lim_{q\to p} k\left(\delta(k-p)\frac{1}{\sinh[(k-q)\pi/2M]}+\delta(k-q)\frac{1}{\sinh[(k-p)\pi/2M]}\right)=0$$

due to the fact that $\sinh x$ is an odd function in x.

From now on, the previously arbitrary mass μ in (1.3) has been set equal to M, the mass of the phonon defined in (3.12). The advantage of this prescription is to make the quntum fluctuations around the soliton, being the elementary phonon excitation, become as independent from each other as possible. Actually, they are the completely free phonons described by plane waves under the approximation made in I.

We find the parameter ν defined in (3.13) as

$$\nu=\tfrac{1}{2}f_0(\beta M)-\tfrac{1}{4}f_1(\beta M) \tag{5.15}$$

where

$$\begin{aligned} f_0(\beta M)&=\frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}k}{\omega_k[\exp(\beta M)_k)-1]}\\ &=T/2M+(1/2\pi)\ln(M/4\pi T)+\gamma/2\pi-[\zeta(3)/2(2\pi)^3](M/T)^2\\ &\quad+\mathrm{O}(M^3/T^3)\\ &\quad(\gamma=0.5772,\ \zeta(3)=1.2021)\end{aligned} \tag{5.16}$$

and

$$\begin{aligned} f_1(\beta M)&=\frac{M^2}{\pi}\int_0^\infty \mathrm{d}k/\omega_k^3[\exp(\beta\omega_k)-1]\\ &=T/4M-1/2\pi+\pi M/12T-[\zeta(3)/(2\pi)^2](M/T)^2+\mathrm{O}(M^3/T^3)\end{aligned} \tag{5.17}$$

are the same as defined by Maki and Takayama (1979). Therefore, in the high-temperature approximation

$$\nu=3T/16M+(4\pi)^{-1}[\ln(M/T)+(\tfrac{1}{2}+\gamma)-\ln 4\pi-\pi^2M/12T]. \tag{5.18}$$

The combination of equations (5.18) and (3.12) does not give us any singular behaviour in M under the weak coupling region $g^2\ll 1$. There only exists a non-physical pole $g^2\nu=-1$ which is irrelevant to the existence of a critical temperature T_c. Formally, the condition

$$g^2\nu(T_c)=1 \tag{5.19}$$

would make the series in $\tilde{H}_2$ divergent and give us a value of T_c. Keeping only the first term in (5.18), we would have

$$T_c^{(1)}=(8\sqrt{2}/3g^2)\tilde{m}_0 \tag{5.20}$$

where

$$\tilde{m}_0^2=m_0^2 2^{-\tilde{g}^2} \tag{5.21}$$

m_0 is the m at $T=0$ and

$$\tilde{g}^2=g^2/(8\pi-g^2). \tag{5.22}$$

In the plane wave approximation, we have instead

$$T_c^{\mathrm{PWA}}=(2\sqrt{2}/g^2)\tilde{m}_0. \tag{5.23}$$

The value of (5.20) or (5.23) is near the same of that found by Maki and Takayama (1979) but it seems to us that it is a false phenomenon.

Because the series of sin $g\hat{\psi}$ and cos $g\hat{\psi}$ in (3.1) have an infinite radius of convergence and the geometrical series obtained after taking pairing contractions systematically as in (3.8) can be summed analytically as follows

$$1-g^2\nu+g^4\nu^2-g^6\nu^3+\ldots=1/(1+g^2\nu).$$

We should trust the function on the right-hand side rather than the individual terms on the left-hand side in spite of the latter having a unit radius of convergence. We think that this attitude is in conformity with many experiences in theoretical physics. So we put the condition (5.19) aside and look again at the expression (3.12) carefully. Actually in our approximation

$$(M/m_0)^{2-g^2/4\pi}=[1+g^2\nu(T/m_0, M/m_0)]^{-1} \tag{5.24}$$

M will decrease monotonically from m_0 as T increases†. Therefore, we claim the non-existence of a critical temperature in the sine–Gordon system. Recently, Zotos and Fowler (1982) obtained the same conclusion by the Bethe-Ansatz approach.

As a final step, we take the ensemble average of the reduced Hamiltonian

$$U\equiv\langle\tilde{H}\rangle=\Omega_s+U' \tag{5.25}$$

where

$$\Omega_s=8M/g^2-M(2\pi+7)(f_0(\beta M)-f_1(\beta M)) \tag{5.26}$$

$$U'=L[-M^2/g+M^2(3f_0(\beta M)-\tfrac{7}{2}f_1(\beta M)+f_3(\beta M)] \tag{5.27}$$

with

$$f_3(\beta M)=\frac{1}{\pi M^2}\int_0^\infty\frac{\omega_k\,\mathrm{d}k}{(\exp(\beta\omega_k)-1)}. \tag{5.28}$$

Since U' is a function of T as well as L, the 'volume' of our system, so Ω_s should be regarded as the thermodynamic potential of the soliton even though it does not depend explicitly on L. Therefore, we define the 'internal energy' of the soliton, i.e., the mass of the soliton as

$$E_s\equiv\Omega_s-T\,\mathrm{d}\Omega_s/\mathrm{d}T. \tag{5.29}$$

Noticing further that M is a function of T and using the leading terms in the expansions of f_0 and f_1, we have

$$E_s\simeq 8M/g^2-(8/g^2)T\,\mathrm{d}M/\mathrm{d}T\simeq 8M/g^2. \tag{5.30}$$

In the last step, we have used the fact that the second term is much smaller than the first one when T is large and $g^2\ll 1$.

6. ϕ^4 system

As this system had been discussed in detail by Takayama and Maki (1979) as well as in I, we will present our result concisely by using the method in the above sections.

† Here we assume $g^2<8\pi$, which is a critical value of g^2 found by Coleman (1975).

Starting from the Lagrangian density

$$\mathscr{L}=\tfrac{1}{2}(\partial\phi/\partial t)^2-\tfrac{1}{2}(\partial\phi/\partial x)^2+\tfrac{1}{4}m_b^2\phi^2-\tfrac{1}{4}g^2\phi^4 \tag{6.1}$$

and using (1.1) and (1.2) we can derive the reduced Hamiltonian as

$$\tilde{H}=H_0+\tilde{H}_1+\tilde{H}_2 \qquad \tilde{H}_1=H_1+\tilde{H}_3 \qquad \tilde{H}_2=H_2+\tilde{H}_4 \qquad (6.2),\ (6.3),\ (6.4)$$

$$H_0=\int \mathrm{d}x[\tfrac{1}{2}(\partial\phi_s/\partial x)-\tfrac{1}{4}m^2\phi_s^2+\tfrac{1}{4}g^2\phi_s^4]+E_0.$$

$$E_0=L\bigg[\int\frac{\mathrm{d}k}{8\pi}\bigg(\frac{2k^2+\mu^2}{\omega_k}\bigg)+\tfrac{1}{2}g^2\bigg(\frac{1}{2\pi}\int \mathrm{d}k\frac{1}{\omega_k}\bigg)^2-m_b^2\frac{1}{4\pi}\int\frac{\mathrm{d}k}{\omega_k}\bigg] \tag{6.5}$$

$$H_1=\int \mathrm{d}x(-\partial^2\phi_s/\partial x^2-\tfrac{1}{2}m^2\phi_s+g^2\phi_s^3)\hat{\psi} \tag{6.6}$$

$$H_2=\sum_k\bigg[\bigg(\frac{k^2}{2\omega_k}+\frac{\omega_k}{2}-\frac{m^2}{4\omega_k}\bigg)c_k^\dagger c_k+\bigg(\frac{k^2}{4\omega_k}-\frac{\omega_k}{4}-\frac{m^2}{8\omega_k}\bigg)(c_k^\dagger c_{-k}^\dagger+c_kc_{-k})\bigg]$$
$$+\sum_{k_1k_2}\bigg(\frac{3g^2}{4L(\omega_{k_1}\omega_{k_2})^{1/2}}\int\phi_s^2(x)\exp[\mathrm{i}(k_1+k_2)x]\,\mathrm{d}x\ (2c_{-k_1}^\dagger c_{k_2}$$
$$+c_{k_1}c_{k_2}+c_{-k_1}^\dagger c_{-k_2}^\dagger)\bigg) \tag{6.7}$$

$$\tilde{H}_3=6g^2\nu\int\phi_s\hat{\psi}\,\mathrm{d}x \tag{6.8}$$

$$\tilde{H}_4=\sum_k\frac{3g^2\nu}{2\omega_k}(2c_k^\dagger c_k+c_k^\dagger c_{-k}^\dagger+c_kc_{-k}) \tag{6.9}$$

where

$$m^2=m_b^2-(3g^2/\pi)\ln(2\Lambda/\mu) \tag{6.10}$$

as discussed in I. ν is the same parameter as that defined in (3.19). The condition

$$\tilde{H}_1=0 \tag{6.11}$$

will lead to the equation for ϕ_s:

$$\mathrm{d}^2\phi_s/\mathrm{d}x^2+\tfrac{1}{2}M^2\phi_s(x)-g^2\phi_s^3(x)=0 \tag{6.12}$$

with

$$M^2=m^2-12g^2\nu. \tag{6.13}$$

Equation (6.12) can also be derived from a variational principle

$$\delta\langle\tilde{H}\rangle/\delta\phi_s(x)=0. \tag{6.14}$$

The solution of equation (6.12) is familiar to us:

$$\phi_s(x)=(M/\sqrt{2}g)\tanh\tfrac{1}{2}Mx. \tag{6.15}$$

For calculating the elementary spectra we follow the same procedure as for the sine–Gordon System. Equations (4.20)–(4.24) remain effective except that the function $u_n(x)$ obeys the equation:

$$(E_n^2-M^2+\mathrm{d}^2/\mathrm{d}x^2+\tfrac{3}{2}M^2\,\mathrm{sech}^2\,\tfrac{1}{2}Mx)u_n(x)=0 \tag{6.16}$$

which differs from (5.1). Equation (6.16) now has the following solutions

$$E_0^2 = 0, \qquad u_0(x) = (3M/8)^{1/2}\,\mathrm{sech}^2\,\tfrac{1}{2}Mx \tag{6.17}$$

$$E_1^2 = \tfrac{3}{4}M^2, \qquad u_1(x) = \tfrac{1}{2}(3M)^{1/2}\,\mathrm{sech}\,\tfrac{1}{2}Mx\,\tanh\tfrac{1}{2}Mx \tag{6.18}$$

$$E_k^2 = M^2 + k^2,$$

$$u_k(x) = N_k[\exp(\tfrac{1}{2}Mx) + \exp(-\tfrac{1}{2}Mx)]^{\mathrm{i}2k/M}F\{-\mathrm{i}2k/M - 2, -\mathrm{i}2k/M + 3 \times |1 - \mathrm{i}2k/M|\exp(-\tfrac{1}{2}Mx)/[\exp(\tfrac{1}{2}Mx) + \exp(-\tfrac{1}{2}Mx)]\} \tag{6.19}$$

where $F(\alpha, \beta|\gamma|z)$ is hypergeometric function. The solution (6.17) just corresponds to the zero mode of (6.15) and

$$u_0(x) \sim \mathrm{d}/\phi_s(x)\,\mathrm{d}x \tag{6.20}$$

as expected. $u_1(x)$ is a bound state. The continuum state $u_k(x)$ has the following asymptotic behaviour.

$$u_k(x) = \begin{cases} N_k\exp(\mathrm{i}kx) & x\to\infty \\ N_k\exp[\mathrm{i}(kx-\eta_k)] & x\to-\infty \end{cases} \tag{6.21}$$

with the phase shift

$$\eta_k = 2[\tan^{-1}(\tfrac{1}{2}M/k) + \tan^{-1}(M/k)]. \tag{6.22}$$

In order to make the calculation tractable, we prefer to use the following approximate representation for $u_k(x)$ (distorted wave approximations):

$$u_k(x) \simeq C_k\exp(\mathrm{i}kx)(k + \mathrm{i}M\tanh\tfrac{1}{2}Mx)(2k + \mathrm{i}M\tanh\tfrac{1}{2}Mx) \tag{6.23}$$

$$C_k^2 = [(4k^2+M^2)(k^2+M^2)L - \tfrac{4}{3}M(4M^2+15k^2)]^{-1} \tag{6.24}$$

$$\tilde{u}_k(p) = (1/2\pi)C_k\{(2\pi(2k^2-M^2)\delta(k-p) - 2\pi(2p+k)/\sinh[\pi(k-p)/M]\}. \tag{6.25}$$

Neglecting the details of calculation, we finally get

$$\nu = \tfrac{1}{2}f_0(\beta M) = T/4M + (1/4\pi)(\ln M/4\pi T + \gamma) + \mathrm{O}(M^2/T^2) \tag{6.26}$$

the same as in I. This implies that the critical temperature does not change in replacing the plane waves by distorted waves in order to describe the phonon states:

$$T_c = (2m_0^3/9\sqrt{3}g^2)\{1 + (9g^2/2\pi m_0^2)[\ln(m_0^2/g^2) - 0.0996]\}. \tag{6.27}$$

On the other hand, the energy of the ϕ^4 system receives a correction due to the phase shifts in continuum states as well as the existence of a bound state. We have

$$U \equiv \langle H\rangle = U' + \Omega_s \tag{6.28}$$

where

$$U' = L[-M^4/16g^2 - M^2(\tfrac{3}{4}f_0(\beta M) - f_3(\beta M)] \tag{6.29}$$

$$\Omega_s = M^3/3g^2 - M(9f_0(\beta M) + 2f_1(\beta M) - \tfrac{35}{3}\tilde{f}_1(\beta M) - 6f_4(\beta M)) - \tfrac{1}{2}M + \tfrac{7}{20}\sqrt{3}M\coth(\tfrac{1}{4}\sqrt{3}\beta M) \tag{6.30}$$

with

$$\tilde{f}_1(\beta M) = \frac{M^2}{\pi}\int_0^\infty \frac{\mathrm{d}k}{\omega_k(4k^2+M^2)[\exp(\beta\omega_k)-1]} = \frac{1}{6\beta M} - \frac{1}{6\sqrt{3}} + \frac{\beta M}{48} + \ldots \tag{6.31}$$

(Takayama and Maki 1979)

$$f_4(\beta M) = \frac{M^4}{\pi}\int_0^\infty \frac{\mathrm{d}k}{\omega_k^3(4k^2+M^2)[\exp(\beta\omega_k)-1]} = \frac{5}{36}\frac{1}{\beta M} + \dots \tag{6.32}$$

The mass of the soliton is defined as before,

$$\begin{aligned} E_s \equiv \Omega_s - T\,\mathrm{d}\Omega_s/\mathrm{d}T \\ &= M^3/3g^2 - \tfrac{1}{2}M + \tfrac{7}{20}\sqrt{3}M\coth(\tfrac{1}{4}\sqrt{3}\beta M) \\ &\quad - T(\mathrm{d}M/\mathrm{d}T)[M^2/g^2 - \tfrac{1}{2} + \tfrac{7}{20}\sqrt{3}\,(\coth\tfrac{1}{4}\sqrt{3}\beta M)] \\ &\quad + \tfrac{21}{20}M(\mathrm{d}M/\mathrm{d}T - M/T)[\exp(\sqrt{3}\beta M/2)]/[\exp(\sqrt{3}\beta M/2)-1]^2. \end{aligned} \tag{6.33}$$

Thus we see

$$E_s|_{T\to 0} = m_0^3/3g^2 + \tfrac{1}{2}m_0(\tfrac{7}{10}\sqrt{3}-1) \tag{6.34}$$

with the correction term not coinciding with that of DHN (Dashen *et al* 1974) and TK (Takayama and Maki 1979).

On the other hand, as $T \to T_c$, because

$$\mathrm{d}M/\mathrm{d}T|_{T\to T_c} = -\sqrt{3}g^2/m_0(M-M_c) < 0 \tag{6.35}$$

we would have

$$E_s|_{T\to T_c} = (1/9\sqrt{3}g^2)m_0^3 - (1/2\sqrt{3})m_0 - (2/27\sqrt{3})(m_0^5/g^4)\left(\frac{\mathrm{d}M}{\mathrm{d}T}\right)_{T\to T_c} \tag{6.36}$$

which will reveal an abrupt rise in the vicinity of $T \to T_c$. $M \to M_c$. We do no know if it is a genuine phenomenon of phase transition or simply an outcome of our approximation.

7. Summary and discussion

We present a method based on the real-time Green function approach to investigate the quantum fluctuations of real scalar fields in a $(1+1)$-dimensional space around the solitons at finite temperatures. As an improved approximation to that in I, the momentum non-conserved Green functions are introduced before we are able to treat the various phonon states in the soliton potentials.

In using the method of the real-time Green function, one has to introduce the pairing approximation for reducing the problem to a mathematically tractable one and this approximation corresponds to the well known Hartree–Fock approximation (Chang 1976). We are fortunate that the pairing approximation here can reduce the original Hamiltonian of S–G system to a quite concise one. Moreover, the decomposition of ϕ into $(\phi_s + \hat{\psi})$ implies, in the language of superfluidity theory, a two-fluid model. The coherent state ϕ_s corresponds to the superfluid component and the incoherent part, the quantised $\hat{\psi}$, to the normal fluid one. It is easy to see that the average square fluctuation is $\langle\phi^2\rangle - \langle\phi\rangle^2 = 2\nu$ under the pairing approximation. Where the property $\langle\hat{\psi}\rangle \equiv 0$ had been used. As we see in the previous sections, ν increases with temperature monotonically. So this picture seems reasonable.

Besides the pairing approximation, we confine our discussion within the one soliton sector and neglect all the multi-soliton sectors. It implies that we have assumed tacitly that the average thermal excitation energy is much smaller than the mass of the soliton, e.g., for S–G system

$$k_B T \ll E_s \sim 8M/g^2.$$

Therefore, when the temperature is not too low, our approximation is legitimate only if $g^2 \ll 1$, i.e., only in the weak coupling limit.

To our knowledge, our treatment is different from that of Maki and Takayama (1979) who used the imaginary Green function approach. Though both are valid only in the weak coupling region, and both are similar at low temepratures, we still expect some different qualtiative behaviour at high temperatures.

For the sine–Gordon system, we find that the masses of the phonons as well as the soliton decrease as the temperatures increases. There is no critical temperature.

On the other hand, for the ϕ^4 system, the phase shifts for continuum states as well as the existence of a bound state do not affect the value of T_c which remains the same as calculated by a plane wave approximation. However, we find that the mass of the soliton reveals an abrupt rise just before the critical temperature, $T \to T_c$. This phenomenon is probably related to the existence of a phase transition in a ϕ^4 system (Chang 1976, Stone 1976, 1977).

As pointed out by Chang (1976), the method of Hartree–Fock approximation cannot correctly describe the nature of a phase transition. We have the same difficulty. For ϕ^4 system, we cannot find the tendency of $M^2 \to 0$ when $T \to T_c$. i.e., we miss the cross-over behaviour characterising the second-order transition. (Maki and Takayama 1979, Takayama and Maki 1979). Nonetheless, further investigation is needed.

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References

Chang S-J 1976 *Phys. Rev.* D **13** 2778
Coleman S 1975 *Phys. Rev.* D **11** 2088
—— 1977 *New Phenomena in Subnuclear Physics, Proc. 14th course of the Int. School Subnuclear Phys. Erice, 1975*, ed A Zichichi (New York: Plenum)
Dashen R F, Hasslacher B and Neveu A 1974 *Phys. Rev.* D **10** 4114, 4130
Gervais J L and Neveu A 1976 *Phys. Rep.* **23C** 240
Jackiw R 1977 *Rev. Mod. Phys.* **49** 681
Maki K and Takayama H 1979a *Phys. Rev.* B **20** 3223
—— 1979b *Phys. Rev.* B **20** 5002
Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* (New York: McGraw-Hill) pp 1649–59
Rajaraman R 1975 *Phys. Rev.* **21C** 229
Rubinstein J 1970 *J. Math.* **11** 258

Stone M 1976 *Phys. Rev.* D **14** 3568
—— 1977 *Phys. Lett.* **67B** 186
Su R-K, Bi P-z and Ni G-j 1983 *J. Phys. A: Math. Gen.* **16** 2445
Takayama H and Maki K 1979 *Phys. Rev.* B **20** 5009
Zotos X and Fowler M 1982 *Phys. Rev.* B **26** 1430